Math 246A Lecture 16 Notes

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1 The Hyperbolic Metric and Integration of Chains

1.1 The hyperbolic metric

Let $z_1, z_2 \in \mathbb{D} = \{z : |z| < 1\}$. Recall that the hyperbolic metric is

$$\rho(z_1, z_2) = \int_\gamma \frac{1}{1 - |z|^2} \, ds.$$

Then $\rho(Tz_1, Tz_2) = \rho(z_1, z_2)$ by the equality case of Pick's theorem. In general, if $f : \mathbb{D} \to \mathbb{D}$ is analytic, then we get

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

so $\rho(f(z_1), f(z_2)) \leq \rho(z_1, z_2)$. So every analytic function is 1-Lipschitz with respect to the hyperbolic metric.

What are geodesics in this metric?

$$\int_{t_1}^{t_2} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{1 - (x(t)^2 + y(t)^2)} \, dt \geq \int_{t_1}^{t_2} \frac{dx/dt}{1 - x(t)^2} \, dt.$$

So the geodesics are circles which intersect the boundary of d at right angles.

Example 1.1. Let's find $B = \{w : \rho(z, w) < 1/6\}$, where $z \in \mathbb{D}$. Since rotation is a Möbious transformation, without loss of generality, $z = r$ with $0 < r < 1$. Let

$$T = \frac{z + r}{1 + rz}$$

send the unit disc to itself and the origin to r. Then $B = T(\{z : |z| < 1/6\})$.

We can also calculate

$$\operatorname{dist}_E(B, \partial D) = 1 - R(1/2) = \frac{1 - r}{3},$$

$$\mathrm{center}_E(B) = \frac{1}{2}(T(1/2) + T(-1/2)) = \frac{3r}{4 - r^2},$$
$$\mathrm{diam}_E(B) = \frac{4(1-r)(1+r)}{4r^2}.$$

Let $\mathbb{H} = \{z + iy : y > 0\}$ be the upper half plane. If we map the unit circle to $\mathbb{H}$ via a Möbius transformation, we can study hyperbolic distance on $\mathbb{H}$.

1.2 Integration of chains

Let Ω be a domain.

Definition 1.1. A **chain** in Ω is a finite formal sum[1] $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ of piecewise C^1 curves in Ω.

Definition 1.2. A chain γ is a **cycle** if γ_j is closed for all j.

Definition 1.3. Equivalence of chains is given by $\gamma \sim \tilde{\gamma}$ if $\tilde{\gamma}$ is obtained from γ by

1. permuting $\{\gamma_1, \dots, \gamma_n\}$
2. reparameterizing a γ_j in increasing fashion
3. subdividing $\gamma = \gamma_1 + \gamma_2$
4. cancelling opposite arcs

Definition 1.4 (reverse orientation gives negative gamma)**.**

Let $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$ with $\gamma_j \neq \gamma_k$ for $j \neq k$ and $a_j \in \mathbb{Z}$, and suppose that f, g are continuous on γ (the set).

Definition 1.5. Integration with respect to γ is defined as

$$\int_\gamma f\,dx + g\,dy := \sum_{j=1}^n a - j \int_{\gamma_j} (f\,dz + g\,dy),$$

$$\int_\gamma F\,dz = \sum_{j=1}^n a_j \left(\int_{\gamma_j} (F\,dx + iF\,dy) \right).$$

Proposition 1.1. *If $\gamma \sim \tilde{\gamma}$, then*

$$\int_\gamma f\,dx + g\,dy = \int_{\tilde{\gamma}} f\,dx + g\,dy.$$

[1] Why do we write this as addition? You should think of these as linear functionals.

Proposition 1.2. *Let γ be a cycle and $f \in C^1$. Let $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. Then*

$$\int_\gamma P\,dx + Q\,dy = 0.$$

Definition 1.6. Let $z \notin \bigcup_j \gamma_j$ (as sets). Then the **winding number** of γ around z is

$$n(\gamma, z) = \sum_{j=1}^{n} a_j n(\gamma_j, z) = \sum_{j=1} a_j \frac{1}{2\pi i} \int_{\gamma_j} \frac{1}{w - z}\,dw.$$

Definition 1.7. Let $\Omega \subseteq \mathbb{C}^*$ be a domain. Ω is **simply connected** if $\mathbb{C}^* \setminus \Omega$ is connected.

Example 1.2. $\mathbb{D}$, $\mathbb{C}$, and $\mathbb{C}^*$ are all simply connected. However, $\mathbb{C} \setminus \{0\}$ is not simply connected. $\mathbb{C} \setminus ([-\infty, 0] \cup [1, \infty))$ is also simply connected.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Then Ω is simply connected if and only if $n(\gamma, z) = 0$ for all cycles $\gamma \subseteq \Omega$ and $z \notin \Omega$.*

Proof. ($\implies$): Let $K = \mathbb{C}^* \setminus \Omega$. Then K is connected. Let γ be a cycle. Then there exists a connected component U of $\mathbb{C}^* \setminus \gamma$ with $K \subseteq U$. So $n(\gamma, z)$ is constant on K. As $\infty \in K$ and $n(\gamma, \infty) = 0$, $n(\gamma, z) = 0$ for all $z \in K$.

($\impliedby$): Assume $\Omega \subseteq \mathbb{C}$ with Ω not simply connected. Then $\mathbb{C}^* \setminus \Omega = A \cup B$, where A, B are closed, nonempty, disjoint sets. Without loss of generality, $\infty \in B$. Then $A \subseteq \{z : |z| \leq M\}$ for some $M < \infty$. Let $\delta > 0$ be such that $\delta < \operatorname{dist}_{\mathbb{C}}(A, B) = \inf\{|z - w| : z \in A, w \in B\}$. Pave $\mathbb{C}$ by closed squares of side length $\delta/\sqrt{2}$ with sides parallel to the axes. Then, for such a square S, $S \cap A \neq \varnothing \implies S \cap B = \varnothing$. Now take $\gamma = \sum_{S \cap A \neq \varnothing} \partial S$. Then γ is a cycle, and $\gamma \subseteq \Omega$; if $a \in \gamma$ but $a \notin \Omega$, then a is on the edge or corner of a square, so we can just change the curve to an equivalent one by cancelling opposite arcs. So

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z - a}\,dz = 1$$

for all $a \in A$. ∎